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Manipulating Gibbs' Phenomenon for Fourier Interpolation

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The Fourier interpolation polynomials of a periodic function with an isolated jump discontinuity at a node exhibit for growing order a Gibbs phenomenon. By a suitable definition of the function value at the jump the over- and undershoots on one side may be minimized. © 1997 Academic Press

Gibbs' phenomenon appears when a periodic function f (of period 2π, say), having a jump discontinuity, is approximated by the partial sums s_n of its Fourier series. Under fast Fourier transformation, f is interpolated instead by a trigonometric polynomial s_n^* of order n (say) in $2n$ nodes $j\pi/n$ $(-n<j\leqslant n)$ [5]. Also in this case, a Gibbs phenomenon may be observed with, however, other overshoot and undershoot values [2]. If the jump occurs at an interpolation node, the shape of the Gibbs phenomenon depends on the value of f at the jump (which would be irrelevant for the partial sums s_n). The purpose of this note is to exhibit the influence of this value on the shape of the Gibbs phenomenon.

As for mean square approximation by s_n, also for interpolation by s_n^* the Riemann localization principle and the uniform convergence in continuity points are valid [7]. In order to study Gibbs' phenomenon, it therefore suffices, without loss of generality, to consider the function f_c defined by

$$f_c(x)=\begin{cases}-1 & \text{for} \quad -\pi<x<0,\\ c & \text{for} \quad x=0,\\ 1 & \text{for} \quad 0<x<\pi,\\ -c & \text{for} \quad x=\pi.\end{cases} \tag{1}$$

As $n\to\infty$, the corresponding interpolation functions given for even n by

$$s_{c,n}^*=\frac{\sin(nx)}{n}\left[\frac{c}{\sin(x)}+\sum_{j=1}^{n-1}\frac{(-1)^j}{\sin(x-j\pi/n)}\right]$$

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converge pointwise to the limit function S_c^* given by

$$S_c^*(x) = \lim_{n \to \infty} s_{c,n}^*\left(\frac{\pi}{n} x\right)$$
$$= c \cdot \frac{\sin(\pi x)}{\pi x} + \frac{\sin(\pi x)}{\pi}$$
$$\times \left\{ \frac{1}{1-x} - \sum_{k=1}^{\infty} \frac{64k(2x-1)}{[(4k+1)^2-(2x-1)^2][(4k-1)^2-(2x-1)^2]} \right\} \tag{2}$$

(This is an easy consequence of the formula in [3, Remark 1, p. 393].) The function S_0^* (the second term of the right member) is odd and it inherits its shape from the functions $s_{0,n}^*$: it assumes exactly one extreme value (alternatingly a maximum and a minimum) in the intervals $]k, k+1[$ while $S_0^*(k) = 1$ $(k \in \mathbb{N})$ (Fig. 1). Roughly stated, the superposition of S_0^* and of the function g defined by

$$g(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & \text{for} \quad x \neq 0, \\ 1 & \text{for} \quad x = 0 \end{cases}$$

in (2) has the following effect: as c increases from 0 to 1, an additional node x_c with $S_c^*(x_c) = 1$ appears and moves from ∞ to 0. This node introduces in the interval $[k, k+1]$ to which it belongs precisely one additional local extremum. This allows us to reduce—to the right of zero—the deviation of the extreme values of S_c^* from the interpolation value 1, at the cost of increasing this deviation to the left of zero.

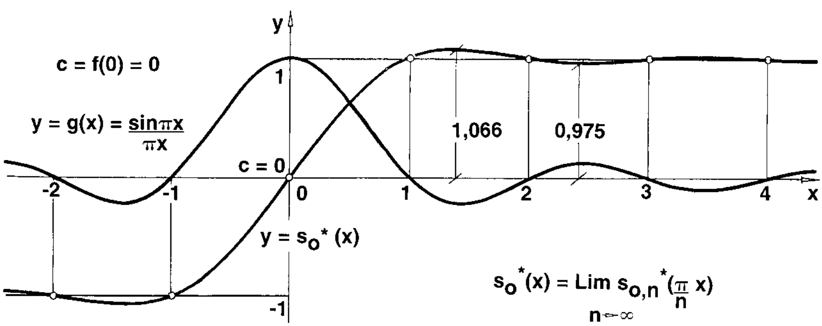

FIGURE 1

These heuristic explanations are made exact by the following theorems.

THEOREM 1. *There is a bijection* $c \mapsto x(c)$ *of* $]0, 1]$ *onto* $[0, \infty[$ *enjoying the following properties*:

(a) $S_c^*(x(c)) = 1$;

(b) $c \mapsto x(c)$ *is monotonically decreasing from* ∞ *to* 0 *on* $]0, 1]$;

(c) *if* $x(c) \in]k, k+1[$ $(k \in \mathbb{N})$, *then* S_c^* *has precisely one local extremum in each of the intervals*

$]j, j+1[$ ($1 \leqslant j < k$; *a maximum for odd j, a minimum for even j*);

$]k, x(c)[$ (*a maximum for odd k, a minimum for even k*);

$]x(c), k+1[$ (*a minimum for odd k, a maximum for even k*);

$]j, j+1[$ ($k+1 \leqslant j < \infty$; *a minimum for odd j, a maximum for even j*);

(d) *if* $x(c) \in [0, 1[$, *then* S_c^* *has precisely one local extremum in each of the intervals*

$]j, j+1[$ ($0 \leqslant j < \infty$; *a minimum for odd j, a maximum for even j*);

(e) *if* $x(c) = k \geqslant 1$, *then* S_c^* *has precisely one local extremum in k (with value* $S_c^*(k) = 1$, *a maximum for odd k, a minimum for even k) and in each of the intervals*

$]j, j+1[$ ($1 \leqslant j < k$; *a maximum for odd j, a minimum for even k*);

($k \leqslant j < \infty$; *a minimum for odd k, a maximum for even k*).

(f) *In any case,* S_c^* *is monotonically increasing in* $[-1, 0]$ *and has a unique extremum in each of the intervals*

$]-j-1, -j[$ ($j \in \mathbb{N}$; *a minimum for odd j, a maximum for even j*).

THEOREM 2. *The function* $x(c)$ *on* $]0, 1]$ *and its inverse function* $c(x)$ *on* $[0, \infty[$ *are real analytic and given by*

$$c(x) = 2 \int_0^1 \frac{t^x \, dt}{(1+t)^2}.$$

We shall present proofs for these theorems along two lines of arguments. The proof of Theorem 1 uses elementary properties of trigonometric polynomials as well as known properties of the functions S_0^*, g and of the values of these functions and their derivatives at the points k ($k \in \mathbb{Z}$). The arguments used for the proof of Theorem 2 at the same time furnish an alternative proof for the assertions (a)–(c) of Theorem 1; they rely on properties of function $\beta(x)$ as listed in [1] and on the partial fraction decomposition of $1/\sin(\pi x)$.

From [2, 3] we retain the following facts.

LEMMA 0. *The function S_0^* is odd and enjoys the following properties*:

(a_0) $S_0^*(k) = 1$ *for* $k \in \mathbb{N}$; $S_0^*(0) = 0$;

(b_0) S_0^* *is monotonically increasing in* $[0, 1]$;

(c_0) S_0^* *has precisely one extremum in each of the intervals* $]j, j+1[$ ($j \in \mathbb{N}$; *a maximum for j odd, a minimum for j even*).

(These statements are consequences of corresponding properties of the interpolating trigonometric polynomials $s_{0,n}^*$ converging to S_0^*.)

In formula (2) S_0^* appears as the sum of a series converging as fast as $\sum_{k=1}^{\infty} 1/k^3$. This representation somewhat blurs its source which is revealed by the formula

$$\frac{\pi}{\sin(\pi x)} S_0^*(x) = \frac{1}{1-x} - 2 \sum_{j=1}^{\infty} \left[\left(\frac{1}{(4j-1)-(2x-1)} - \frac{1}{(4j-1)+(2x-1)} \right) - \left(\frac{1}{(4j+1)-(2x-1)} - \frac{1}{(4j+1)+(2x-1)} \right) \right]$$

$$= \sum_{j=1}^{\infty} (-1)^{j+1} \left(\frac{1}{j-x} + \frac{1}{j+x} \right) \qquad (3)$$

$$= -\beta(x) - \beta(-x), \qquad (4)$$

where

$$\beta(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+x} \qquad [1, 8.372.1].$$

Fixing some $k \in \mathbb{N}$ and disregarding the term $1/(k-x)$, differentiation of the series (3) furnishes a series converging uniformly in $x \in]k - \frac{3}{4}, k + \frac{3}{4}[$. The same is then true for differentiation of S_0^*, since $\sin(\pi x)/(k-x)$ is continuously differentiable. As a consequence, S_0^* is continuously differentiable on $\mathbb{R}$.

Considering the derivative of S_0^* at nonnegative $x \notin \mathbb{N}$, we obtain

$$S_0^{*\prime}(x) = \cos(\pi x) \sum_{j=1}^{\infty} (-1)^{j+1} \left[\frac{1}{j-x} + \frac{1}{j+x} \right] + \frac{\sin(\pi x)}{\pi} \sum_{j=1}^{\infty} (-1)^{j+1} \left[\frac{1}{(j-x)^2} - \frac{1}{(j+x)^2} \right].$$

For $x=0$ this gives

$$S_0^{*\prime}(0) = 2 \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} = 2 \log 2.$$

For $x \to k \in \mathbb{N}$ observe that, by $\sin(\pi x) = (-1)^{k+1} \sin(\pi(k-x))$, we have

$$\begin{aligned} &\lim_{x \to k} \frac{\cos(\pi x) + \sin(\pi x)/(\pi(k-x))}{k-x} \\ &\quad = \lim_{x \to k} \frac{\cos(\pi x) + (-1)^{k+1} \sin(\pi(k-x))/(\pi(k-x))}{k-x} \\ &\quad = \lim_{x \to k} \left[-\pi \sin(\pi x) + (-1)^{k+1} \left(\frac{\sin(\pi(k-x))}{\pi(k-x)} \right)' \right] \Big/ (-1) \\ &\quad = 0. \end{aligned}$$

Using the continuity of $S_0^{*\prime}$ we obtain

$$\begin{aligned} S_0^{*\prime}(k) &= (-1)^{k+1} \left[\frac{1}{k} + 2 \sum_{j=1}^{\infty} \frac{(-1)^j}{k+j} \right] &(5) \\ &= (-1)^{k+1} \sum_{j=0}^{\infty} (-1)^j \left[\frac{1}{k+2j} - \frac{2}{k+2j+1} + \frac{1}{k+2j+2} \right] \\ &= (-1)^{k+1} \sum_{j=0}^{\infty} \frac{2}{(k+2j)(k+2j+1)(k+2j+2)}. &(6) \end{aligned}$$

From (5) and (6) we deduce that (for the upper estimate suppose that $k > 2$)

$$S_0^{*\prime}(k) = 2 \left[\log 2 - \left(\sum_{j=1}^{k-1} \frac{(-1)^{j+1}}{j} + \frac{(-1)^{k+1}}{2k} \right) \right], \tag{7}$$

$$\frac{1}{2(k+1)^2} \leqslant |S_0^{*\prime}(k)| \leqslant \frac{1}{2(k-2)^2}. \tag{8}$$

Since $S_c^* = c \cdot g + S_0^*$ (2), we list without proof the properties of the function g as compared to those of S_0^* established in Lemma 0 (Fig. 1).

LEMMA 1. *The function g is even and enjoys the following properties*:

(a_1) $g(k) = 1$ *for* $k = 0$; $g(k) = 0$ *for* $k \in \mathbb{N}$;

(b_1) g *is monotonically decreasing in* $[0, 1]$;

(c_1) g *has precisely one extremum in each of the intervals* $]j, j+1[$ ($j \in \mathbb{N}$; *a minimum for j odd, a maximum for j even*).

Note that $g'(k) = (-1)^k/k$ for $k \neq 0$. Comparing the values of the functions S_0^*, g, and of their derivatives we deduce from (6), (7), and (8):

$$\operatorname{sign} g(x) = \begin{cases} \operatorname{sign}(S_0^*(x) + 1) & \text{for} \quad x \leqslant -1, \\ -\operatorname{sign}(S_0^*(x) - 1) & \text{for} \quad x \geqslant 1, \end{cases} \tag{9}$$

$$\operatorname{sign} g'(k) = -\operatorname{sign} S_0^{*\prime}(k) \qquad \text{for} \quad k \in \mathbb{N}, \tag{10}$$

$$|g'(k)| > |S_0^{*\prime}(k)|. \tag{11}$$

We now study the effect of adding $c \cdot g$ to S_0^* for $0 < c \leqslant 1$. By (9) we see that in $]-\infty, -1]$ only the "hills" of the graph of S_0^* are heightened and the "valleys" are deepened. Also by (9) it is to be expected that in $[1, \infty[$ this effect is reversed and, in fact, also the type of the extrema may be changed. While in $[-1, 0]$ the function S_c^* obviously increases from -1 to c, the situation in $[0, 1]$ may depend on c.

For $k \in \mathbb{N}$, we observe that by (10), (11), and by the linearity in c of $(c \cdot g)'(k)$ there has to exist a solution $c_k \in]0, 1[$ of the equation

$$c_k \cdot g'(k) + S_0^{*\prime}(k) = 0;$$

in fact,

$$c_k = k \sum_{j=0}^{\infty} \frac{2}{(k+2j)(k+2j+1)(k+2j+2)} \qquad \text{by (6).} \tag{12}$$

For formal reasons we put $c_0 = 1$.

LEMMA 2. (i) $1 > c_1 > \cdots > c_k > c_{k+1} > \cdots > 0$;

(ii) $\lim_{k \to \infty} c_k = 0$.

Proof. The last inequality of (i) and assertion (ii) are evident (cf. also (12)), the first inequality of (i) follows from (5) or (11). In order to show $c_k > c_{k+1}$ put $u_k = (-1)^{k+1} S_0^{*\prime}(k)$ and observe

$$u_k \geqslant u_{k+1} \qquad \text{by (6)},$$

$$u_k = \frac{2}{k(k+1)(k+2)} + u_{k+2} \qquad \text{by (6)},$$

$$\begin{aligned} c_k - c_{k+2} &= k u_k - (k+2) u_{k+2} && \text{by (12)}, \\ &= 2\left(\frac{1}{k+1} - \frac{1}{k+2} - u_{k+2}\right) \\ &= 2u_{k+1} && \text{by (5)}. \end{aligned}$$

We conclude

$$c_k = 2 \sum_{j=0}^{\infty} u_{k+1+2j}$$

from which our assertion follows. ∎

Theorem 1 now follows from Propositions 1 and 2 below.

PROPOSITION 1. *For* $c \in \,]c_{k+1}, c_k[$ $(k \in \mathbb{N})$, *resp. for* $c \in \,]c_1, 1]$ $(k=0)$ *there exists a unique* $x(c) \in \,]k, k+1[$ *such that*

(a′) $S_c^*(x(c)) = 1$;

(b′) $x(c)$ *decreases monotonically from* $k+1$ *to* k *on* $]c_{k+1}, c_k[$;

(c′) S_c^* *has precisely one local extremum in each of the intervals as mentioned in Theorem* 1(c), (*resp. Theorem* 1(d)).

Proof. The derivative of S_c^* in the positive nodes $j \in \mathbb{N}$ is

$$\begin{aligned} S_c^{*\prime}(j) &= c\,\frac{(-1)^j}{j} + (-1)^{j+1} u_j \\ &= \frac{(-1)^j}{j}(c - j u_j) \\ &= \frac{(-1)^j}{j}(c - c_j). \end{aligned}$$

If $c \in \,]c_{k+1}, c_k[$, then

$$c - c_j \begin{cases} < \;\; c_k - c_j \;\leqslant 0 & \text{for} \quad j \leqslant k, \\ > c_{k+1} - c_j \geqslant 0 & \text{for} \quad j \geqslant k+1. \end{cases}$$

Consequently, the sign of $S_c^{*\prime}(j)$ is alternating for $j \leqslant k$ (and the same as that of $S_0^{*\prime}(j)$) as well as for $j \geqslant k+1$ (and the opposite as that of $S_0^{*\prime}(j)$), while it is the same for $j=k$ and $j=k+1$. Since $S_c^*(j)=1$ and since S_c^* is continuously differentiable, there has to exist a $x(c) \in]k, k+1[$ such that $S_c^*(x(c))=1$, and there must exist at least two local extreme values (a maximum and a minimum) instead of one in this interval. The number of local extreme values of S_c^* in $]0, n[$ $(n \geqslant k+1)$ is therefore at least n.

Suppose now there were an additional point $\bar{x}_c$ in some interval $]j, j+1[$ $(j \in \mathbb{N}_0)$ satisfying $S_c^*(\bar{x}_c)=1$. Then there would have to exist a second one $\bar{\bar{x}}_c \in]j, j+1[$ (possibly coinciding with $\bar{x}_c$) satisfying $S_c^*(\bar{\bar{x}}_c)=1$. This would introduce two additional local extreme values for S_c^* in $]j, j+1[$. Since S_c^* is the pointwise limit of a sequence of trigonometric polynomials (2), there would have to be a trigonometric polynomial

$$h_n(x)=s_{c,n}^*\left(\frac{\pi x}{n}\right)$$

of even order n with period $2n$ such that the graph of h_n has at least $n+1$ local extreme values between 0 and n. Note that h_n satisfies the symmetry relation

$$h_n(x-n)=s_{c,n}^*\left(\frac{\pi x}{n}-\pi\right)=-h_n(x).$$

The number of local extreme values of h_n in a period would then be at least $2n+2$ which is impossible.

Consider now the case $c_1<c<1=c_0$. Then $S_c^{*\prime}(1)<0$, so there exists $x(c) \in]0, 1[$ such that $S_c^*(x(c))=1$ and there has to exist a local maximum of S_c^* in $]x(c), 1[$. This last assertion is trivial for $c=1$ and $x(c)=0$. The same reasoning as before asserts the uniqueness of $x(c)$ in these cases.

In order to show assertion (b′), for simplicity suppose k to be odd (a similar reasoning applies for even k). Then

$$S_c^*(x)-1\begin{cases} >0 & \text{for} \quad x \in]k, x(c)[\,, \\ <0 & \text{for} \quad x \in]x(c), k+1[\,, \end{cases}$$

and $g(x)<0$ in $]k, k+1[$. Let $c_k>\bar{c}>c>c_{k+1}$. Then

$$S_{\bar{c}}^*(x)=S_c^*(x)+(\bar{c}-c)\, g(x)<S_c^*(x) \qquad \text{for} \quad x \in]k, k+1[$$

and therefore $x(\bar{c})<x(c)$. ∎

Proposition 2. *For $k \in \mathbb{N}$ and $c=c_k$ let $x(c_k)=k$. Then $S_{c_k}^*(x(c_k))=1$ and $S_{c_k}^*$ has a local extremum in k (a maximum for k odd, a minimum for k even) and precisely one local extreme value in any of the intervals as mentioned in Theorem* 1(e).

Proof. If $c = c_k$ $(k \in \mathbb{N})$ then $S_{c_k}^{*\prime}(k) = 0$ while $S_{c_k}^{*\prime}(k-1)$ and $S_{c_k}^{*\prime}(k+1)$ are nonzero and of different sign. A reasoning as above shows that for $k \geqslant 2$ there have to exist local extreme values in $]k-1, k[$ and $]k, k+1[$, and in k (with modifications for $k = 1$ as stated in Theorem 1). A counting argument as above shows that $S_{c_k}^{*}(x) \neq 1$ for all positive $x \notin \mathbb{N}$. ∎

Having established the existence of the order-reversing bijection $c \mapsto x(c)$ of $]0, 1]$ onto $[0, \infty[$ we may consider its inverse $x \mapsto c(x)$ given by the solution of the equations

$$c(x) \cdot g'(x) + S_0^{*\prime}(x) = 0 \qquad \text{for} \quad x = k \in \mathbb{N},$$

$$c(x) \cdot g(x) + S_0^{*}(x) = 1 \qquad \text{for} \quad 0 < x \notin \mathbb{N}.$$

Explicitly, we get

$$c(k) = c_k \qquad \text{for} \quad x = k \in \mathbb{N},$$

$$c(x) = \frac{\pi x}{\sin(\pi x)} - \frac{x}{1-x} + \sum_{j=1}^{\infty} \frac{64 j x(2x-1)}{[(4j+1)^2 - (2x-1)^2][(4j-1)^2 - (2x-1)^2]}$$
$$\text{for} \quad 0 < x \notin \mathbb{N}.$$

The function $c(x)$ and its inverse function $x(c)$ are real analytic as may be seen entering the second line of arguments leading to a proof of Theorem 2. Continuing on (4) observe that by [1, 1.422.3] for $x \notin \mathbb{N}$

$$\begin{aligned} \frac{\pi}{\sin(\pi x)} &= \frac{1}{x} + \sum_{j=1}^{\infty} (-1)^j \left(\frac{1}{x-j} + \frac{1}{x+j} \right) \\ &= -\frac{1}{x} + \sum_{j=0}^{\infty} (-1)^j \left(\frac{1}{j+x} - \frac{1}{j-x} \right) \\ &= -\frac{1}{x} + \beta(x) - \beta(-x), \end{aligned}$$

and, therefore,

$$\begin{aligned} S_c^{*}(x) - 1 &= c \, \frac{\sin(\pi x)}{\pi x} + S_0^{*}(x) - 1 \\ &= \frac{\sin(\pi x)}{\pi x} \left(c - x\beta(x) - x\beta(-x) + 1 - x\beta(x) + x\beta(-x) \right) \\ &= \frac{\sin(\pi x)}{\pi x} \left(c + 1 - 2x\beta(x) \right). \end{aligned} \tag{13}$$

The function $x\beta(x)$ is meromorphic with simple poles in $-\mathbb{N}$, therefore S_c^* is an entire function.

By (13) we have $S_c^*(x)=1$ iff $x \in \mathbb{N}$ or $c = 2x\beta(x)-1$. Let us therefore define $c(x)=2x\beta(x)-1$ and consider $x>-1$. By [1, 3.251.7] (where the range of validity should read $\operatorname{Re}\mu>-1$ (cf. [4, Section 68, (2), p. 169]) we have

$$\begin{aligned} c(x) &= 4\int_0^1 \frac{t^{2x+1}}{(1+t^2)^2}\,dt \\ &= 2\int_0^1 \frac{t^x\,dt}{(1+t)^2} \qquad \text{for} \quad x>-1, \\ c'(x) &= 2\int_0^1 \frac{t^x \log(t)}{(1+t^2)^2}\,dt < 0 \qquad \text{for} \quad x>-1, \\ \lim_{x\to\infty} c(x) &= 0, \\ \lim_{x\to -1} c(x) &= \infty \\ c(0) &= 1. \end{aligned} \tag{14}$$

Remark 1. For $n \in \mathbb{N}$ by [1, 8.375.2] and in agreement with (7) and (12) we have

$$\begin{aligned} c(n) &= 2n\beta(n)-1 \\ &= 2n(-1)^{n+1}\left[\log(2)+\sum_{j=1}^{n-1}\frac{(-1)^j}{j}\right]-1; \end{aligned}$$

in particular,

$$\begin{aligned} c(1) &= 2\log(2)-1 \approx 0.386, \\ c(2) &= 3-4\log(2) \approx 0.227, \\ c\left(\frac{1}{2}\right) &= 2\int_0^1 \frac{\sqrt{t}\,dt}{(1+t)^2} \\ &= \frac{\pi}{2}-1 \approx 0.571. \end{aligned}$$

Since

$$\beta(x+1) = \frac{1}{x} - \beta(x)$$

we get

$$c\left(n+\frac{1}{2}\right)=(-1)^n\,\frac{(2n+1)\pi}{2}+2(2n+1)\sum_{k=1}^{n}\frac{(-1)^{n+k}}{2k-1}-1;$$

in particular

$$c\left(\frac{3}{2}\right)=5-\frac{3\pi}{2}\approx 0.288.$$

Remark 2. As apparent from (14) the function $c(x)$ is monotonically decreasing on the interval $]-1,\infty[$. The real analytic bijective mapping $c=c(x)\Leftrightarrow x=x(c)$ may therefore be extended to a decreasing mapping $]0,\infty[\,\leftrightarrow\,]-1,\infty[$.

Remark 3. From (13) one may infer that

$$|S_c^*(z)|\leqslant C\cdot e^{\pi\,|\mathrm{Im}(z)|}\qquad \text{for}\quad z\in\mathbb{C}.$$

By the Paley–Wiener theorem [6, VI.4 and I.13], S_c^* is the Fourier transform of a distribution with compact support. In the case of $c=0$, this Fourier transform representation is in fact given by

$$\begin{aligned}S_0^*(x)&=\frac{i}{2\pi}\lim_{\varepsilon\to 0}\left\{\int_{-\pi}^{-\varepsilon}e^{-ix\xi}\cot\left(\frac{\xi}{2}\right)d\xi+\int_{\varepsilon}^{\pi}e^{-ix\xi}\cot\left(\frac{\xi}{2}\right)d\xi\right\}\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(x\xi)\cot\left(\frac{\xi}{2}\right)d\xi.\end{aligned}$$

Remark 4. In order to inspect some numerical evidence put $c=c(3/2)\approx 0.288$. For the evaluation of

$$S_0^*(x)=-\frac{\sin(\pi x)}{\pi}\,(\beta(x)+\beta(-x)),$$

one can use the fast convergent series

$$\beta(x)=\sum_{j=0}^{\infty}\frac{j!\,2^{-j-1}}{x(x+1)\cdots(x+j)};$$

cf. [1, 8.372.3]. A computation of values of S_c^* to three decimals with stepsize 0.01 reveals the local extreme values:

$$S_c^*(1.19) \approx 1.008$$

$$S_c^*(1.75) \approx 0.996$$

$$S_c^*(2.52) \approx 1.013$$

$$S_c^*(3.50) \approx 0.986$$

$$S_c^*(4.49) \approx 1.013$$

from which one may infer that

$$|S_c^*(x) - 1| \leqslant 0.014 \qquad \text{for all} \quad x \geqslant 1,$$

but

$$S_c^*(-1.40) \approx -1.128$$

Note that to the right of 1 the maximal deviation of S_c^* from 1 appears with abscissa ≈ 3.50.

Suppose the intention is now to reduce the Gibbs phenomenon by an appropriate choice of $c = \mathbf{c}$ as much as possible in the following sense: to the right of the first point in which the graph of $S_{\mathbf{c}}^*$ crosses the level 1 the deviation of $S_{\mathbf{c}}^*$ from 1 should be as small as possible. Numerical evidence seems to indicate that this is obtained for $\mathbf{c} \approx 0.265$ with a corresponding value of $x(\mathbf{c}) \approx 1.66$ (Fig. 2). In $x_1 \approx 1.22$ the function $S_{\mathbf{c}}^*$ attains an absolute maximum $S_{\mathbf{c}}^*(x_1) \approx 1.012$, in $x_2 \approx 3.50$ a local minimum $S_{\mathbf{c}}^*(x_2) \approx 0.988$, and $S_{\mathbf{c}}^*(x_1) \geqslant S_{\mathbf{c}}^*(x) \geqslant S_{\mathbf{c}}^*(x_2)$ for $x \geqslant 1$. $S_{\mathbf{c}}^*$ is still monotonically increasing in $[-1, 1]$ and assumes in $x_3 \approx -1.40$ an absolute minimum value $S_{\mathbf{c}}^*(x_3) \approx -1.123$.

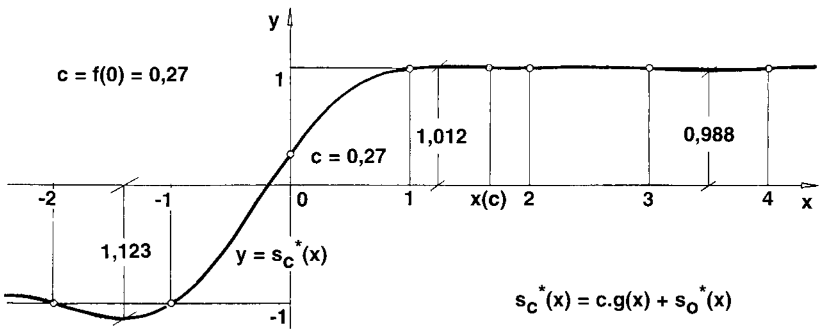

FIGURE 2

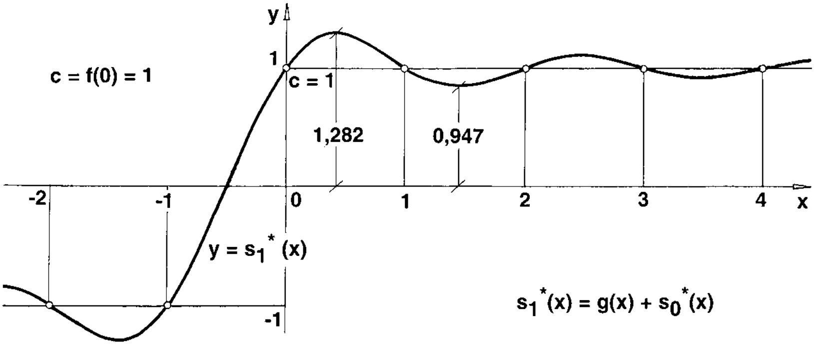

FIGURE 3

As a consequence, defining the value of a function f at a jump node x_0 to be

$$f(x_0)=\frac{f(x_0^+)+f(x_0^-)}{2}+\mathbf{c}\cdot\frac{f(x_0^+)-f(x_0^-)}{2}$$

the Gibbs phenomenon for Fourier interpolation with $2n$ equidistant nodes is reduced to an eventual deviation of less than 1.2% of $(f(x_0^+)-f(x_0^-))/2$ from the function value $f(x)$ to the right side of x_0, at the cost of increasing the absolute undershoot to the left side of x_0 to approximately 12.3% of $(f(x_0^+)-f(x_0^-))/2$. This should be compared with the maximal overshoot of 17.9% of half the jump size for the classical Gibbs phenomenon and of 28.2% of half the jump size for $c=1$. In this last case (Fig. 3) the function S_1^* assumes the same form as if the jump occurred outside of a node, e.g., in an irrational multiple of π instead of in 0.

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